Laplace-Spectra for Shape Recognition

Shape Analysis of Medical Data

Dr. Martin Reuter

Department of Mechanical Engineering
Massachusetts Institute of Technology

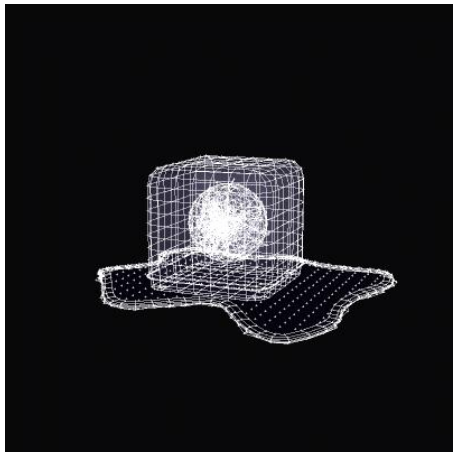
Overview

Comparing and Identifying Shape

Martin Reuter

Question: What is “Shape” and what is “similar”?

- Is shape just the outer shell of an object (B-Rep)?
- What if the object contains cavities?

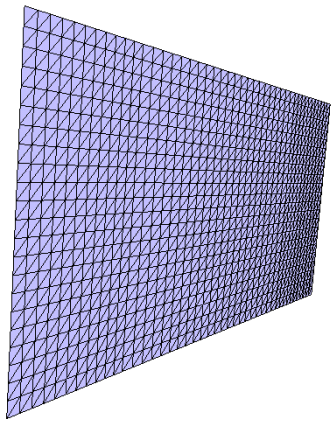


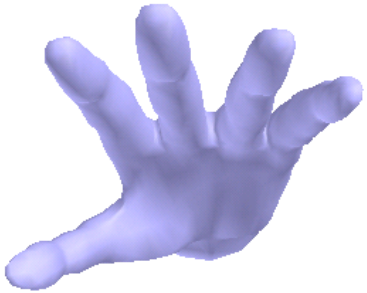
- Shape should be invariant under translation and rotation (congruence)!
- How about scaling invariance (sometimes)?

Comparing and Identifying Shape

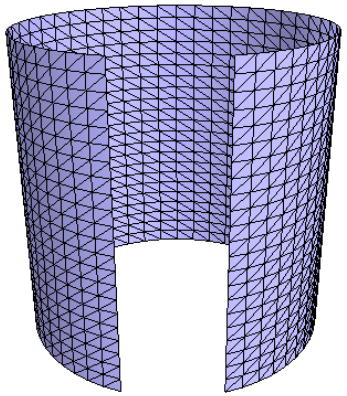
Martin Reuter

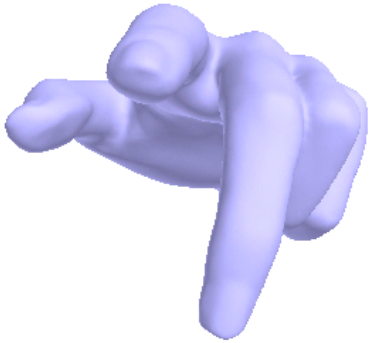
- Isometry invariance?





$\Updownarrow$ $\Updownarrow$

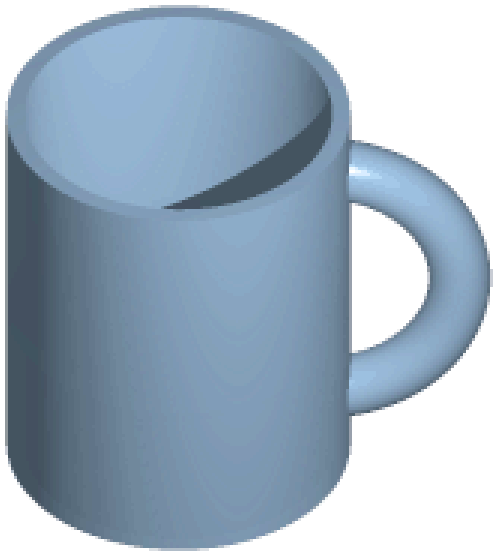


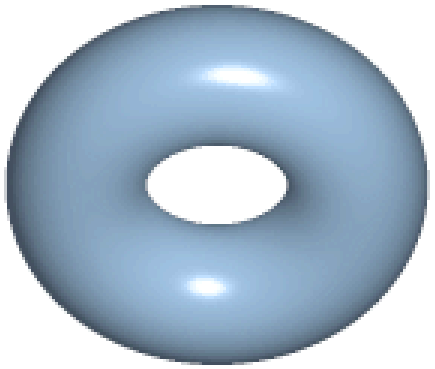


Comparing and Identifying Shape

Martin Reuter

- Homotopy invariance?





http://en.wikipedia.org/wiki/Topology

Different Representations and Parameters

Not only do spacial parameters differ, but:

- surfaces and solids can be given in many *different representations* (e.g. parametrized surfaces, 3d polygonal models, implicitly defined surfaces ...).

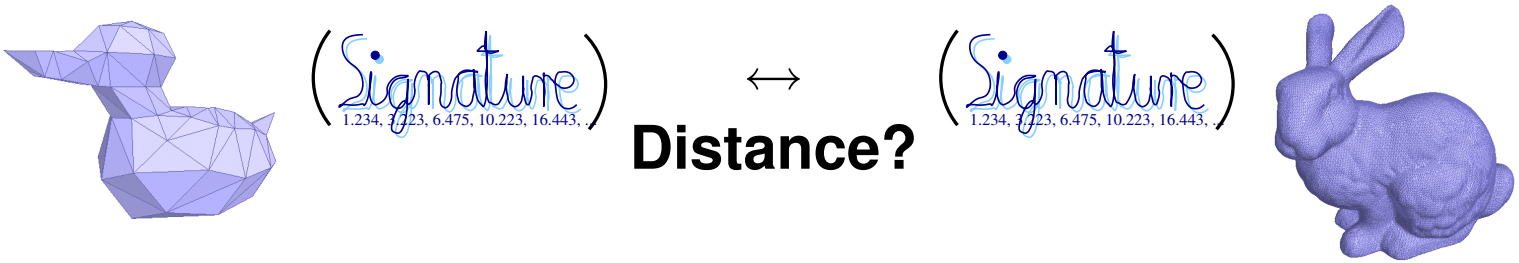
Goal

To find a method for shape identification and comparison that is independent of the given representation of the object.

Identification and Comparison

Martin Reuter

Shape-Matching



0.) Prior alignment, scaling of the objects: normalization, registration

1.) Computation of a simplified representation (Signature, Shape-Descriptor)

2.) Comparison of the signatures, distance computation to measure similarity

Disadvantages of current methods

Disadvantages of current methods

- Simplification too strong (too many objects with identical signatures)
- Missing invariance, complex pre-processing
- Complicated comparison of signatures (e.g. graph based signatures)
- Only special representations (Voxels, Triangulations)
- Depending on supplementary information / context

New Signature: Shape-DNA

Martin Reuter

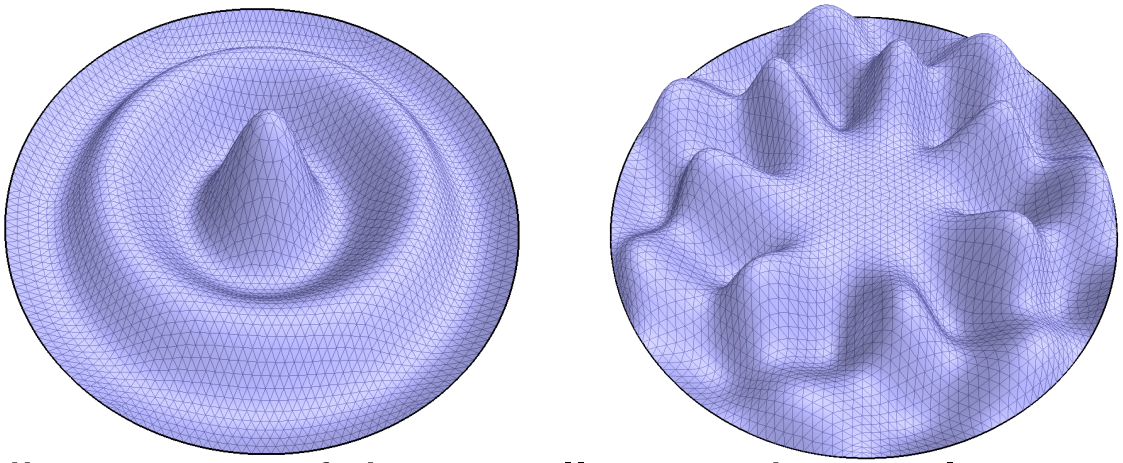
- Invariant under translation, rotation and (where required) scaling
- No registration / normalization necessary
- Surfaces & solids (even with cavities), arbitrary genus
- Independent of representation
- Isometry invariant
- Simple distance computation of the signatures
- No user interaction

Sound of a drum

Martin Reuter

Sound is influenced by the material and the shape

- A surface can be understood as an oscillating membrane (fixed at the boundary)



- We use the n-dim vector of the smallest n eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of the Laplace operator Δ as the signature:

(Shape-DNA)

Definition of the Laplace-Spectrum

Martin Reuter

For a real-valued function f on a Riemannian manifold M

Definition

Helmholtz Equation (**Laplacian Eigenvalue Problem**):

$$\Delta f = -\lambda f$$

Solution: Eigenfunctions f_i with corresponding family of eigenvalues (**Spectrum**):

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \uparrow +\infty$$

Here Laplace-Beltrami Operator: $\Delta f := div(grad\ f)$

Laplace-Spectrum

Laplace-Beltrami Spectrum

- Above definition holds for 2D-surfaces as well as for 3D-solids (independent of representation)

Dirichlet Boundary Condition

Function is fixed $f \equiv 0$ on the boundary of M

Neumann Boundary Condition

Derivative in normal direction is fixed $\frac{\partial f}{\partial n} \equiv 0$ on the boundary of M

Computation of the Spectrum

Martin Reuter

1. Variational formulation of the Helmholtz Equation:

$$\iint \varphi \Delta f \; d\sigma \quad = -\lambda \iint \varphi f \; d\sigma$$

2. Discretization with the finite element method (up to cubic form functions F_m for triangles and voxels and quadratic for tetrahedra)

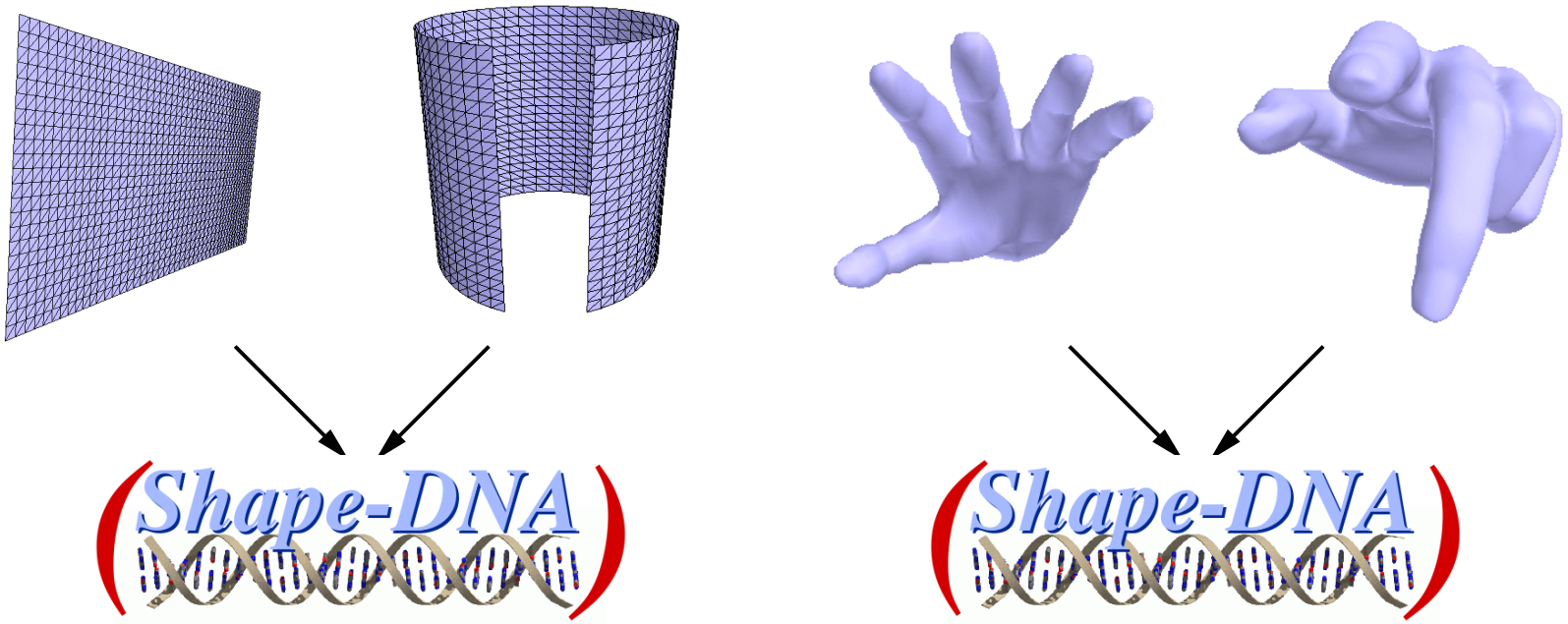
$$f = \sum_{m=1}^{n} u_m F_m$$

3. Solution of the resulting general eigenvalue problem $Au = \lambda Bu$
(sparse and sym.: Lanczos from ARPACK).

Isometric Invariant

Martin Reuter

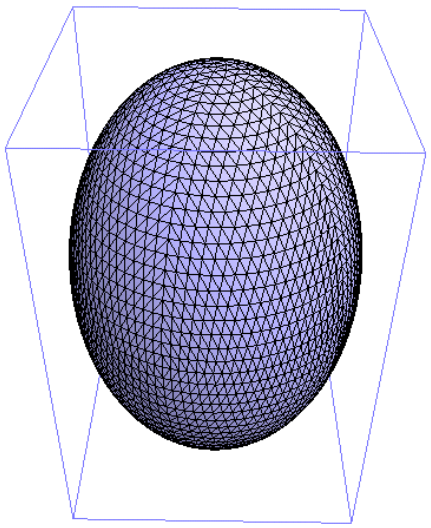
Isometric objects have the same spectrum!



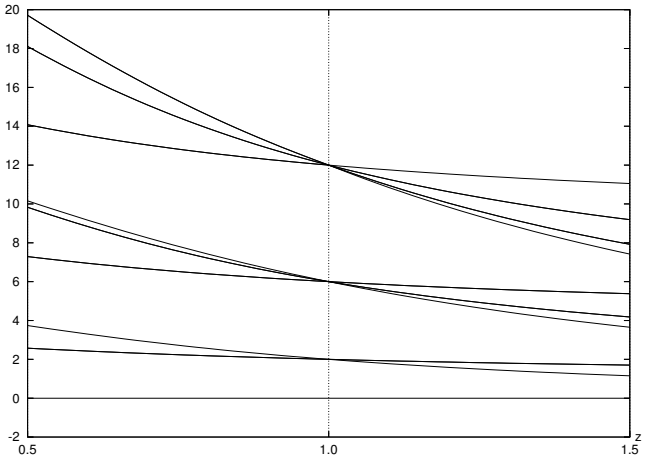
- Spectrum is independent of object's spacial position.
- If M is scaled by s, signature is scaled by s^{-2}.

Continuous Dependency on Deformation

Martin Reuter



The spectrum depends **continuously** on the shape.

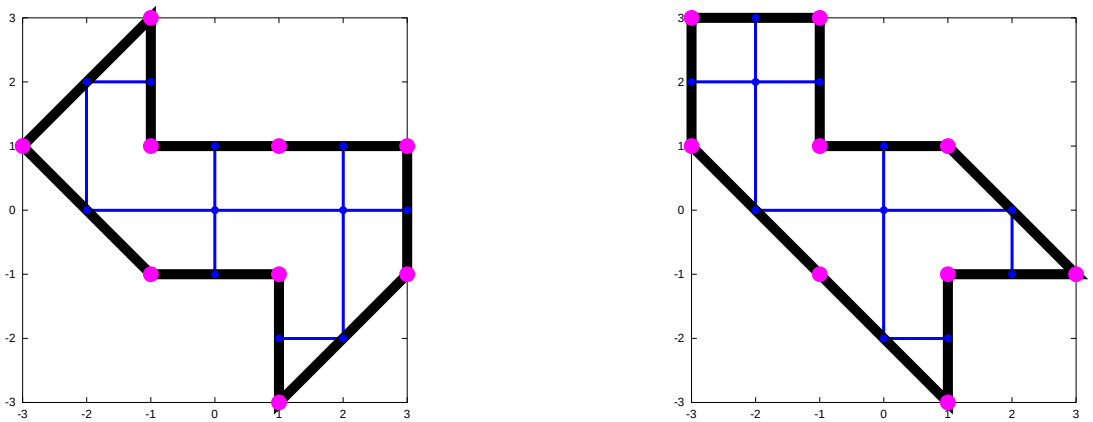


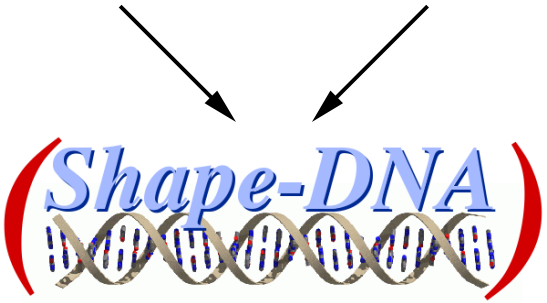
GWW Drums

Martin Reuter

The spectrum does not completely determine the isometry class. Isospectral but not isometric (Gordon, Webb, Wolpert - 1992) drums:

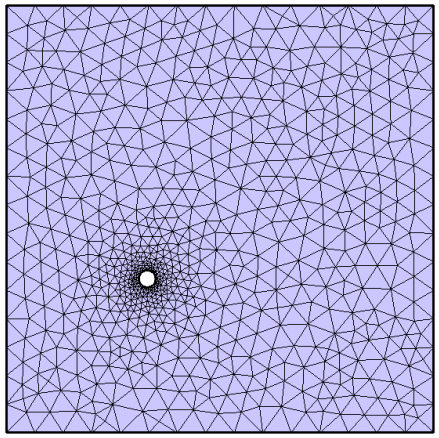
- rare
- concave in 2D
- only pairs





Unwanted Holes - Neumann Condition

Martin Reuter



- Dirichlet spectrum strongly depends on boundary
- Unwanted holes (e.g. missing triangles) have large influence
- Change in topology changes spectrum discontinuously

Unwanted Holes - Neumann Condition

Martin Reuter

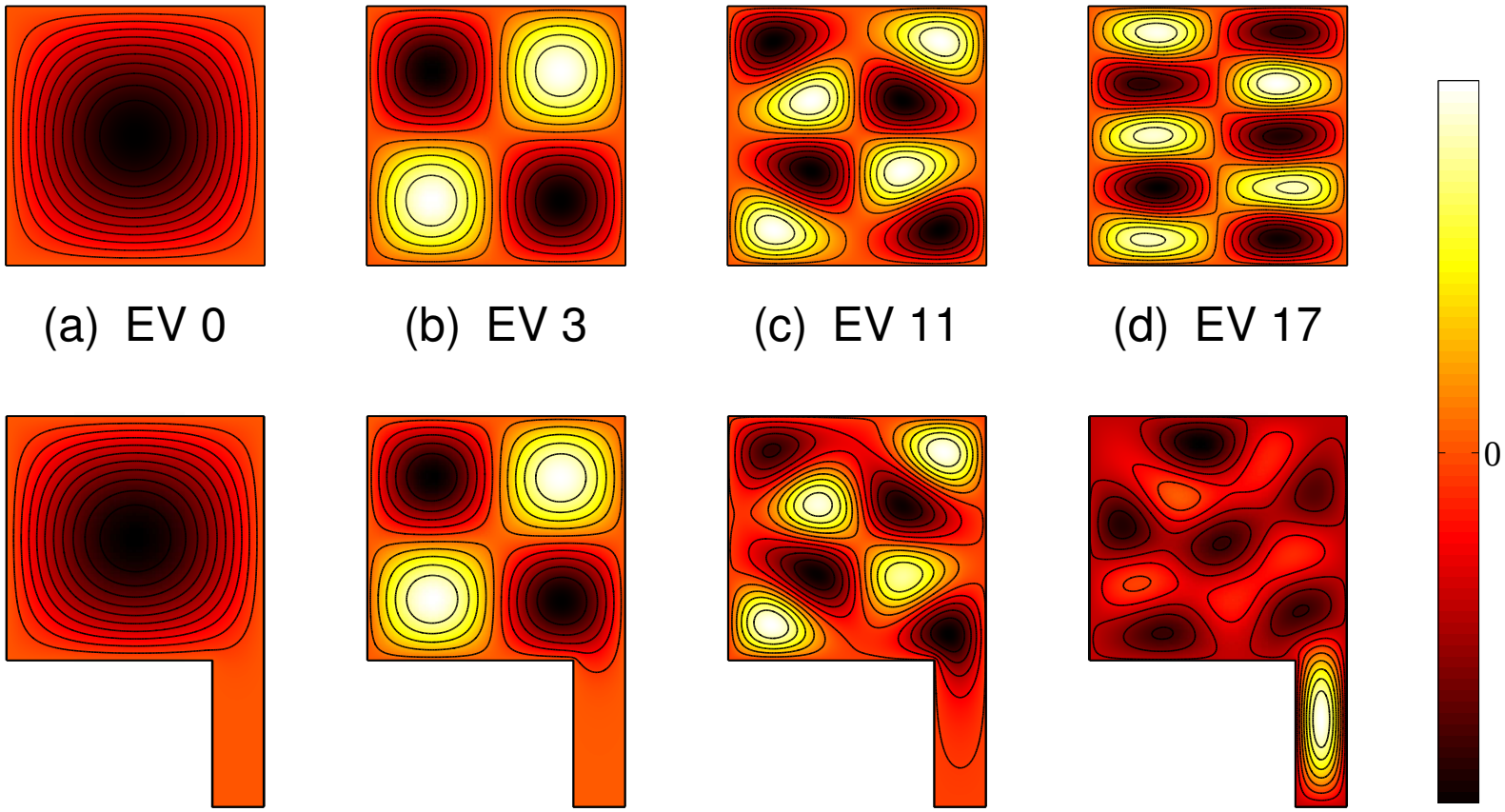
- Neumann spectrum does not change as drastically:

	Sq Dir	*SqH Dir*	*Sq Neu*	*SqH Neu*
λ_1	19.739	24.730	00.000	00.000
λ_2	49.348	49.530	09.870	09.831
λ_3	49.348	61.051	09.870	09.841
λ_4	78.957	87.482	19.739	19.710
λ_5	98.696	99.390	39.478	39.336

Sq: Square, *SqH*: Square with hole,
Dir: Dirichlet, *Neu*: Neumann boundary condition

Early Sensibility - Neumann Condition

Martin Reuter



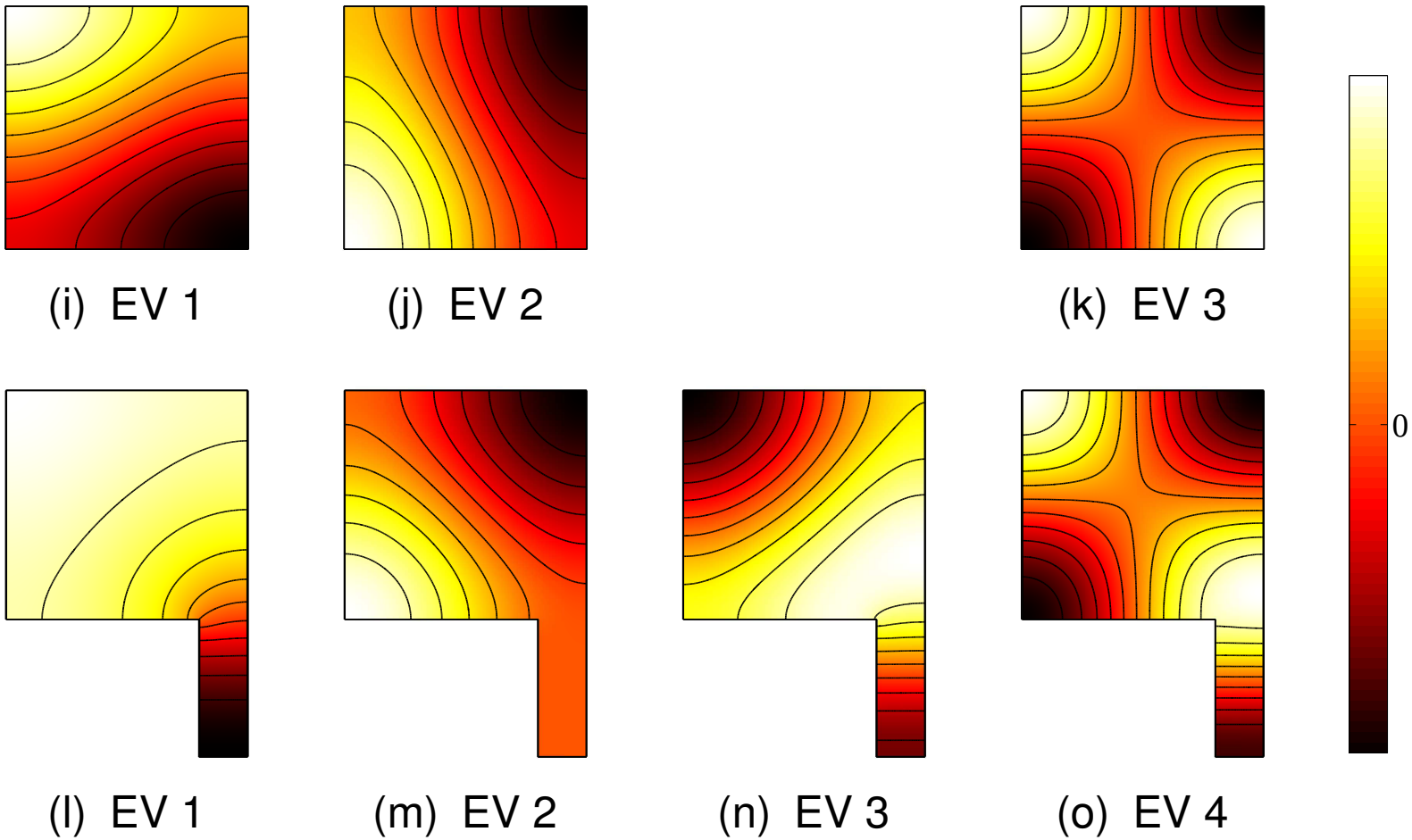
(a) EV 0 (b) EV 3 (c) EV 11 (d) EV 17

(e) EV 0 (f) EV 3 (g) EV 11 (h) EV 17

Dirichlet boundary conditions.

Early Sensibility - Neumann Condition

Martin Reuter



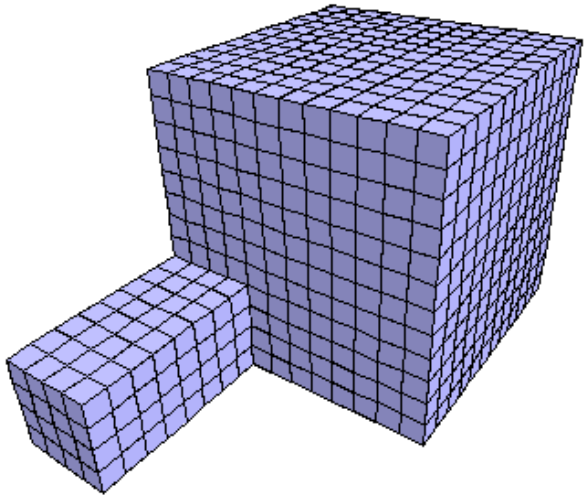
(i) EV 1 (j) EV 2 (k) EV 3

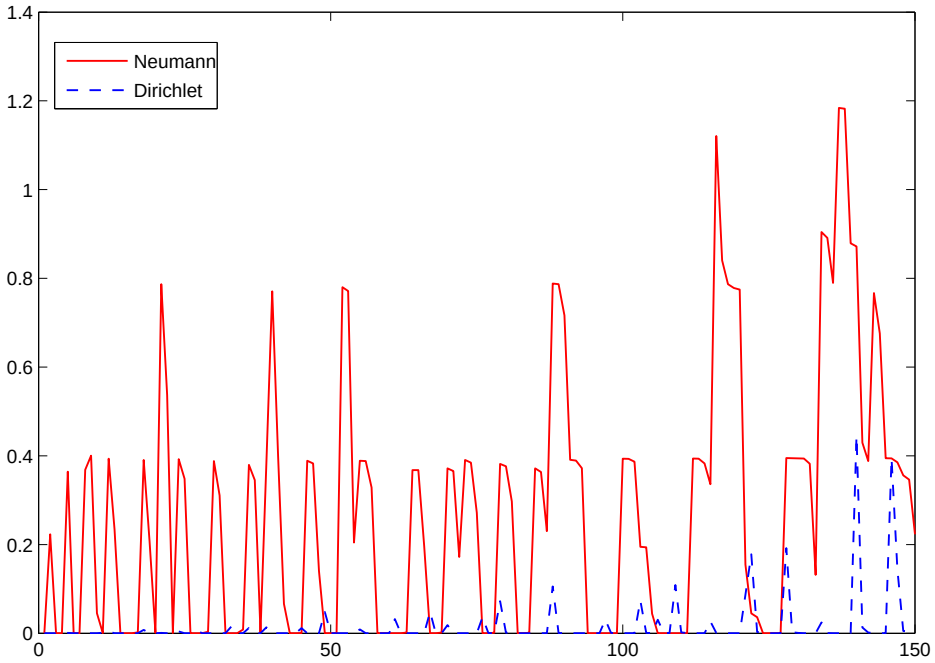
(l) EV 1 (m) EV 2 (n) EV 3 (o) EV 4

Neumann boundary conditions.

Early Sensibility - Neumann Condition

Martin Reuter





The first 150 eigenvalues of the cube with tail subtracted from the eigenvalues of the cube for the Dirichlet and Neumann case

Geometric and Topological Information

Further geometric and topological information is contained in the Spectrum (Heat-Trace Expansion):

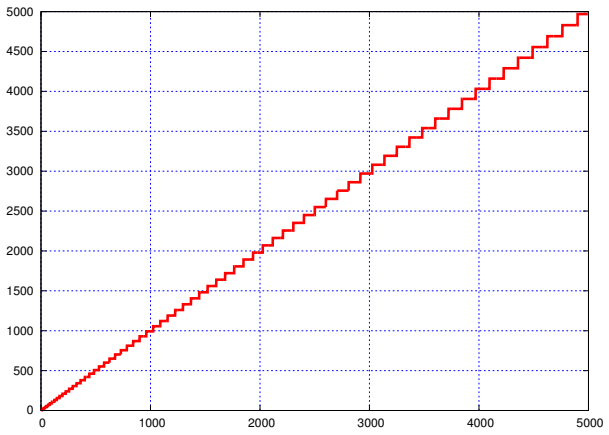
- Riemannian volume
- Riemannian volume of the boundary
- Euler characteristic for closed 2D manifolds
- Number of holes for planar domains

It is possible to extract this data numerically from the beginning sequence of the spectrum
(Reuter 2006 - first 500 eigenvalues).

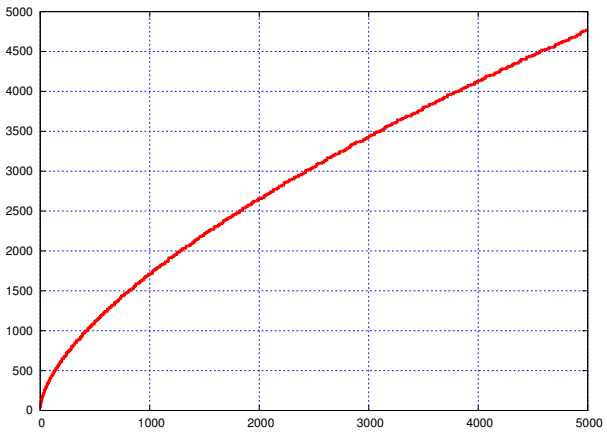
Weyl's Formular

Martin Reuter

EW 2D-Sphere



EW 3D-Cube



Theorem (Weyl - 1912)

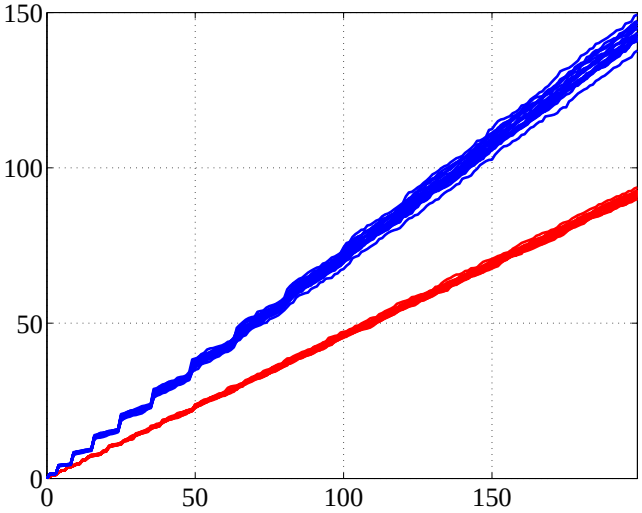
$$\lambda_n \sim \frac{4\pi}{\mathrm{area}(D)}\, n \qquad \textit{for } d = 2 \textit{ and } n \to \infty$$

$$\lambda_n \sim \left(\frac{6\pi^2}{\mathrm{vol}(D)}\right)^{\frac{2}{3}} n^{\frac{2}{3}} \qquad \textit{for } d = 3 \textit{ and } n \to \infty.$$

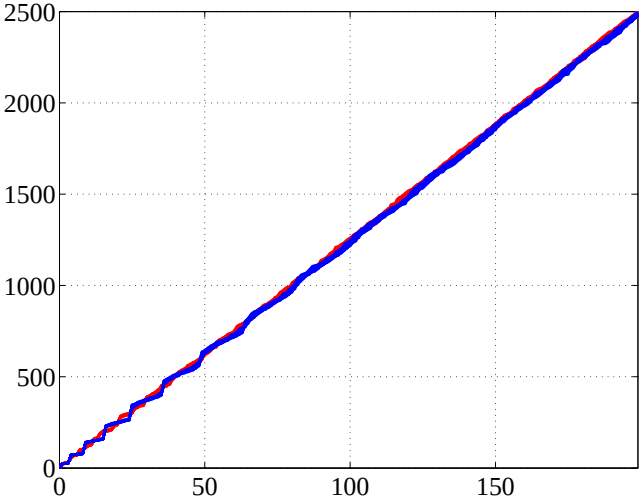
Normalization

Martin Reuter

Two classes of spectra of spheres and ellipsoids with noise
blue : noisy spheres , red : noisy ellipsoids



unnormalized



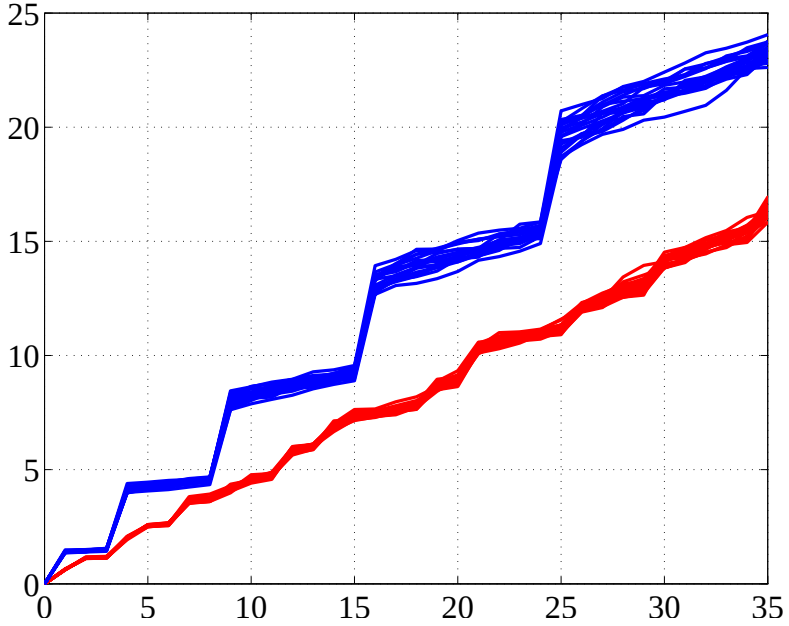
area normalized

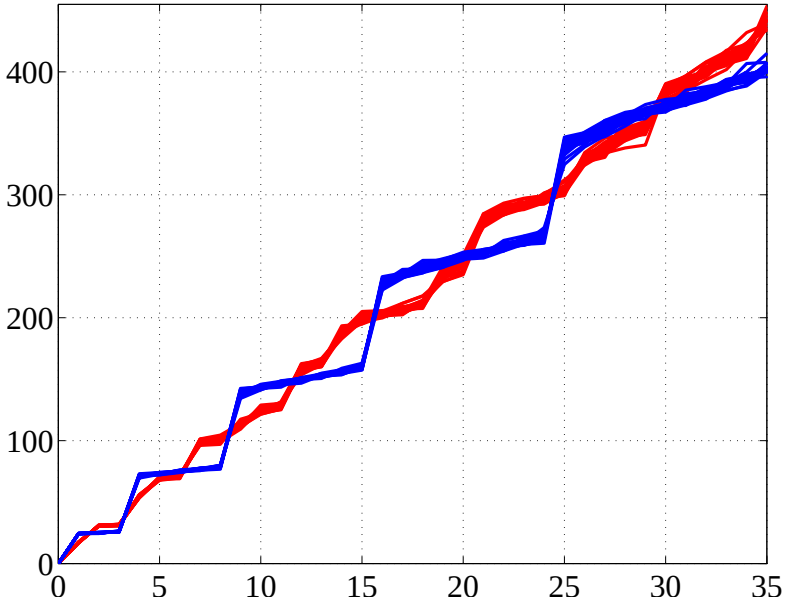
- Shape analysis results depend on chosen normalization
- Unnormalized: Mainly differences in area/volume

Normalization

Martin Reuter

Zoom-ins on the two spectra classes:



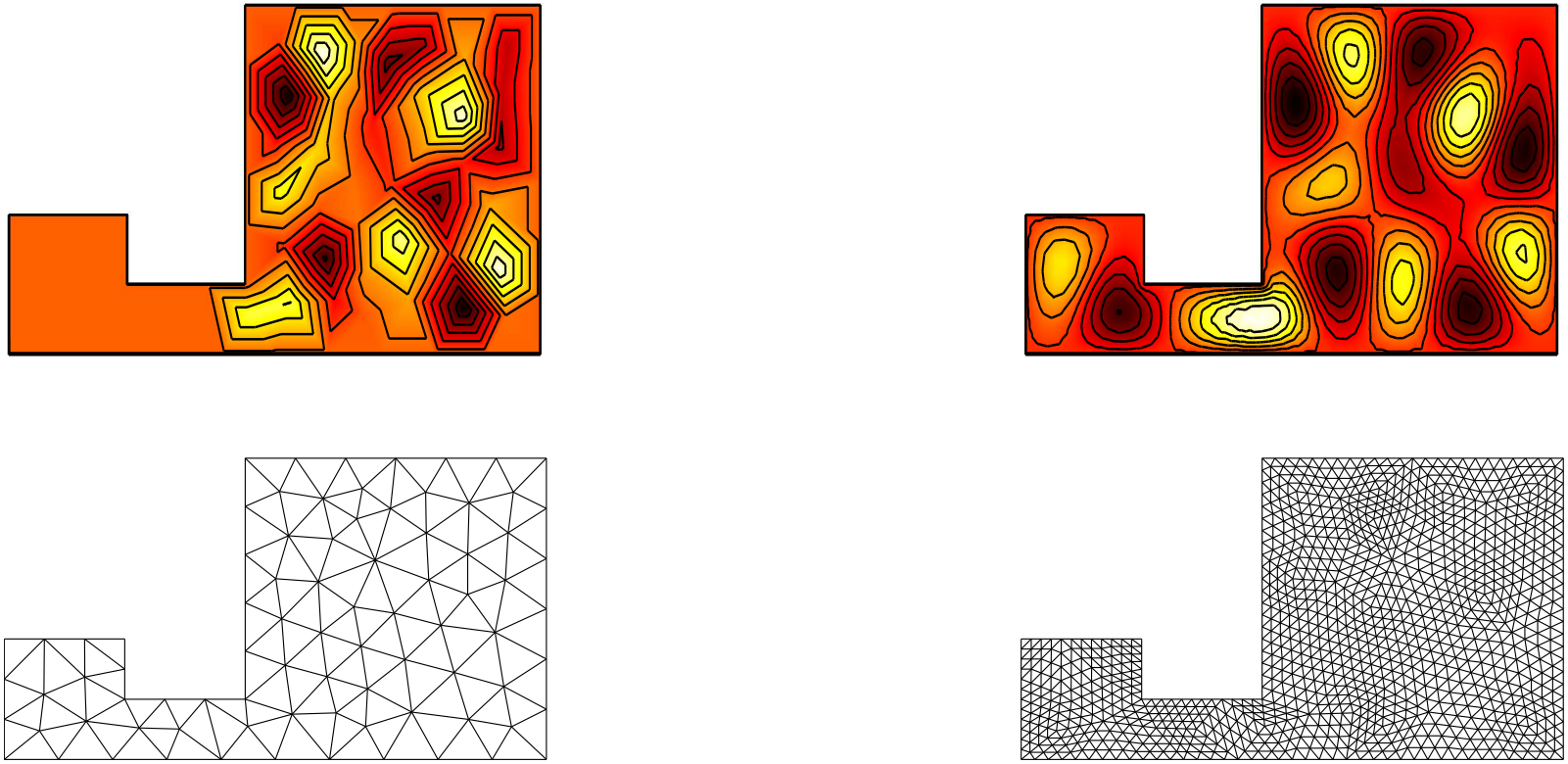


blue : noisy spheres , red : noisy ellipsoids

- Area/volume normalization shows if additional **shape differences** exist.

Influence of Discretization

Martin Reuter

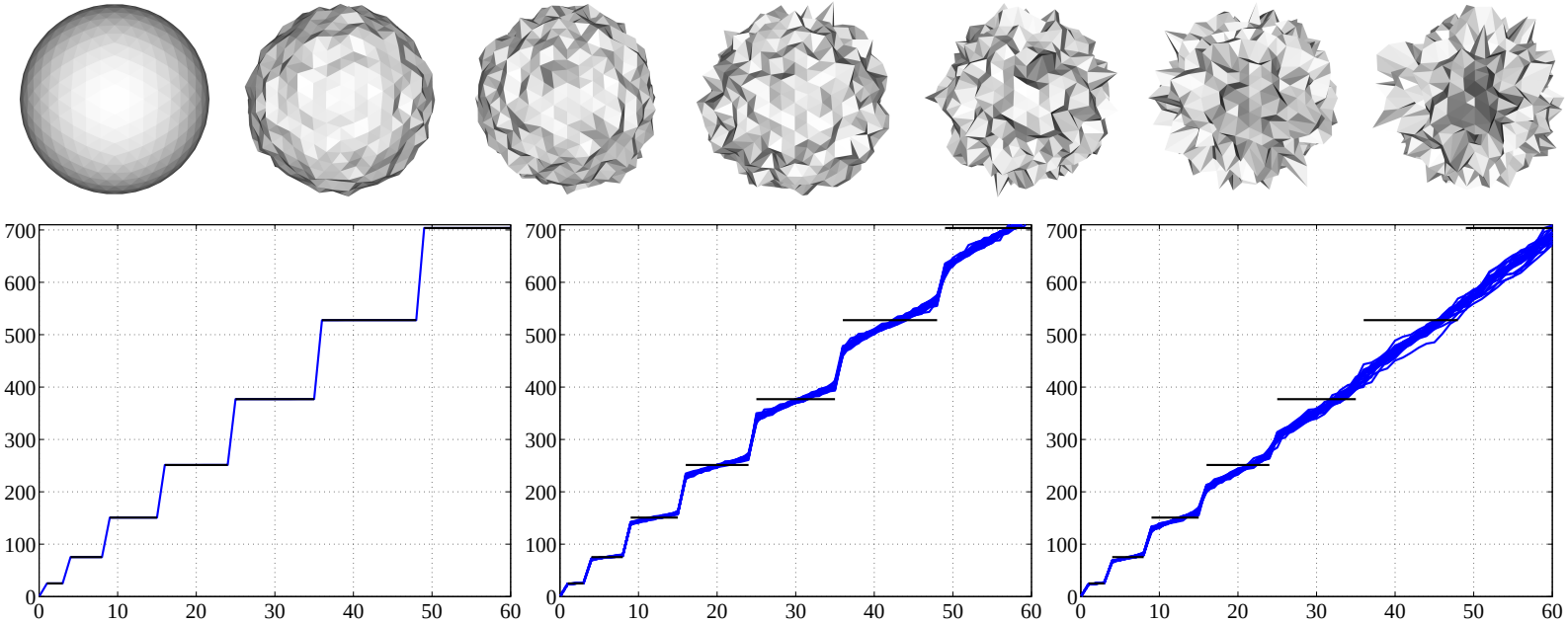


Eigenmode 19 for Dirichlet boundary conditions for different mesh refinements. Thin structures may be overlooked for coarse discretizations (left).

Influence of Noise

Martin Reuter

Spectra of spheres, different noise levels (area normalized)



- Essential to have identical noise levels (increased noise $\rightarrow$ increased surface area)
- Violating this assumption may yield detection of noise level differences instead of shape differences (if shapes are similar or noise is huge).

Normalization and Distance Computation

1. Computation of the first n eigenvalues (Shape-DNA)

2. Normalization (optional)
   a) Surface area normalized
   b) Volume normalized

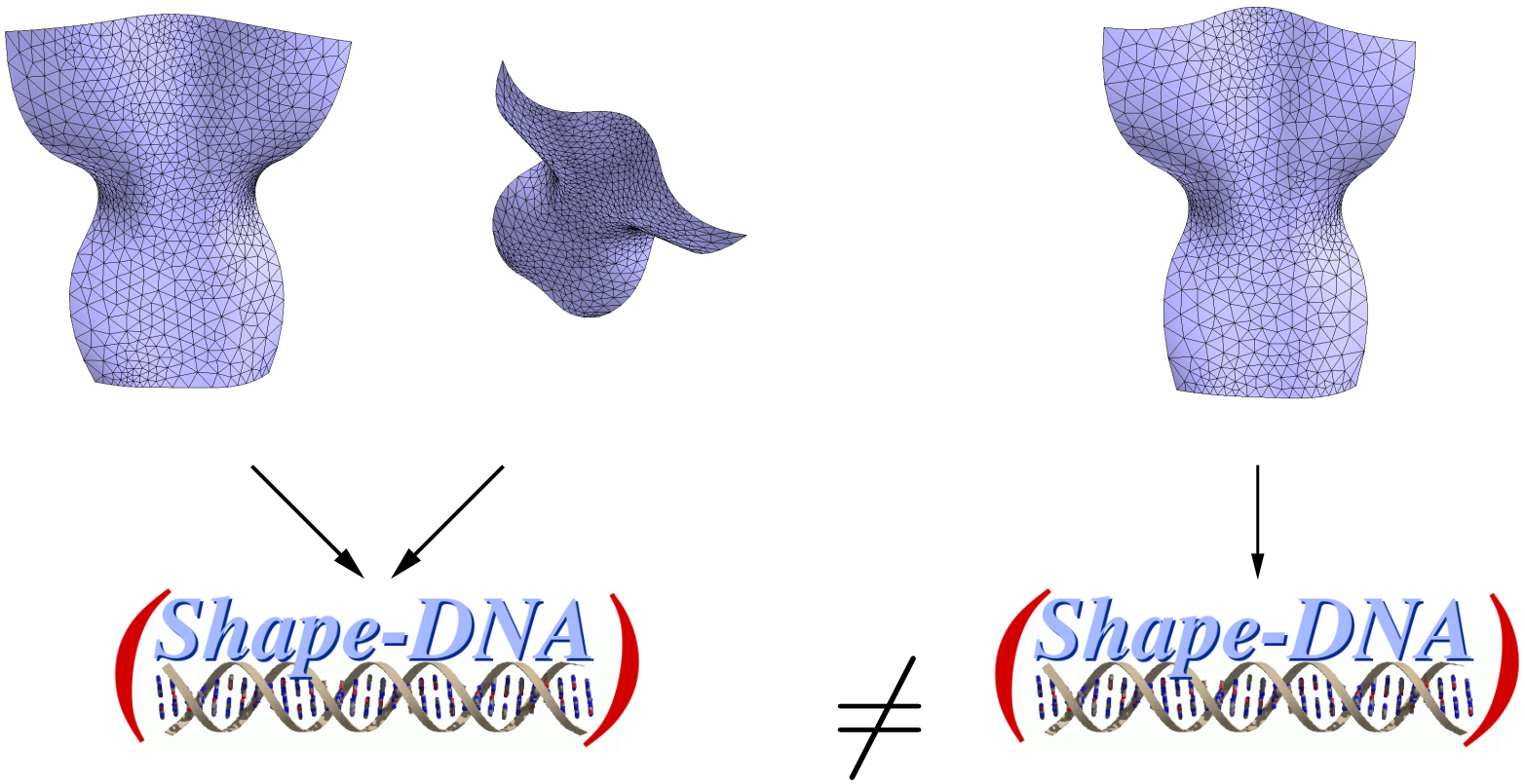
3. Distance computation of the Shape-DNA (n-dim vector)
   a) **Euclidean distance** $\leftarrow$
   b) Another p-norm
   c) Hausdorff distance
   d) Correlation ...

Identification in DB, Copyright protection, Quality assessment

Martin Reuter

Different representations ⇒

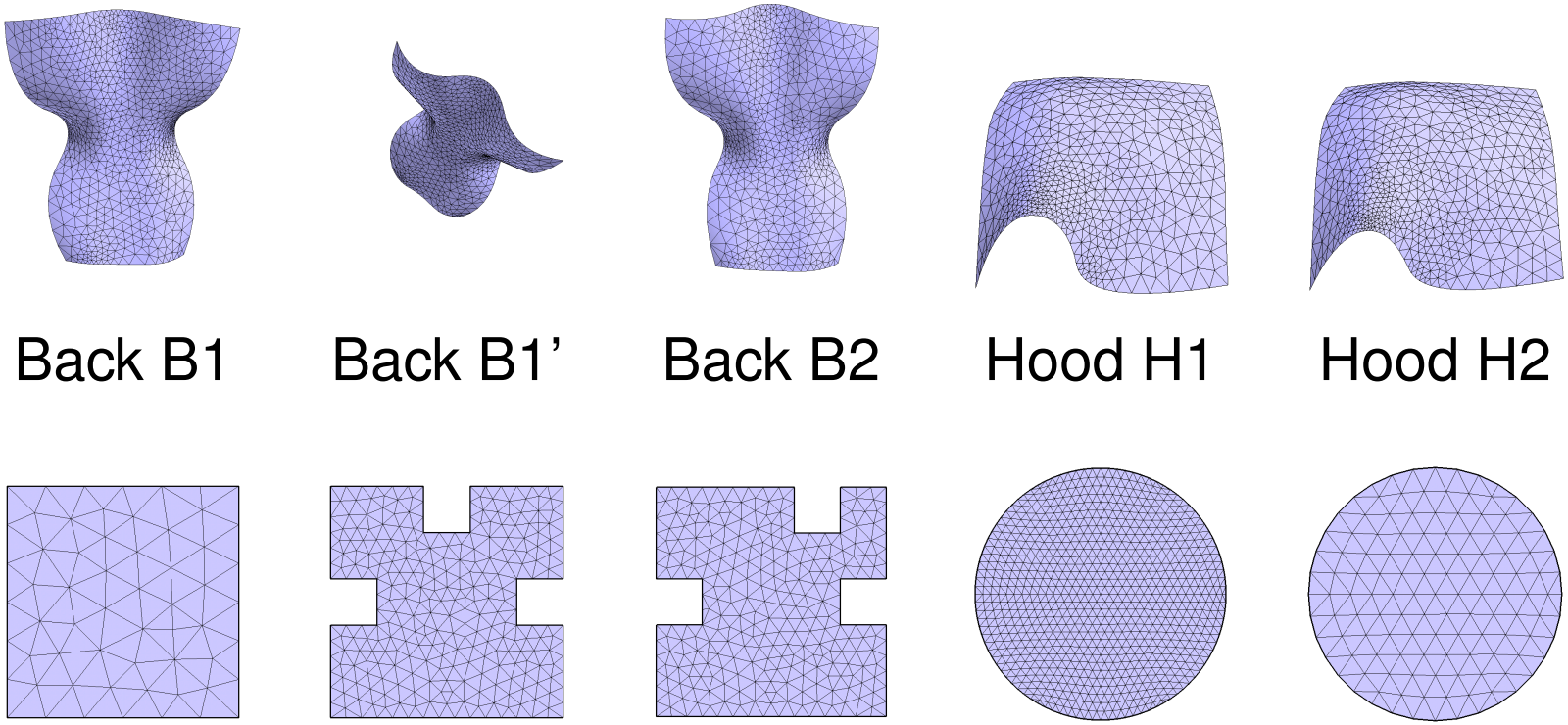
- challenging to identify a protected object
- challenging to retrieve a specific object from DB



Similarity Detection

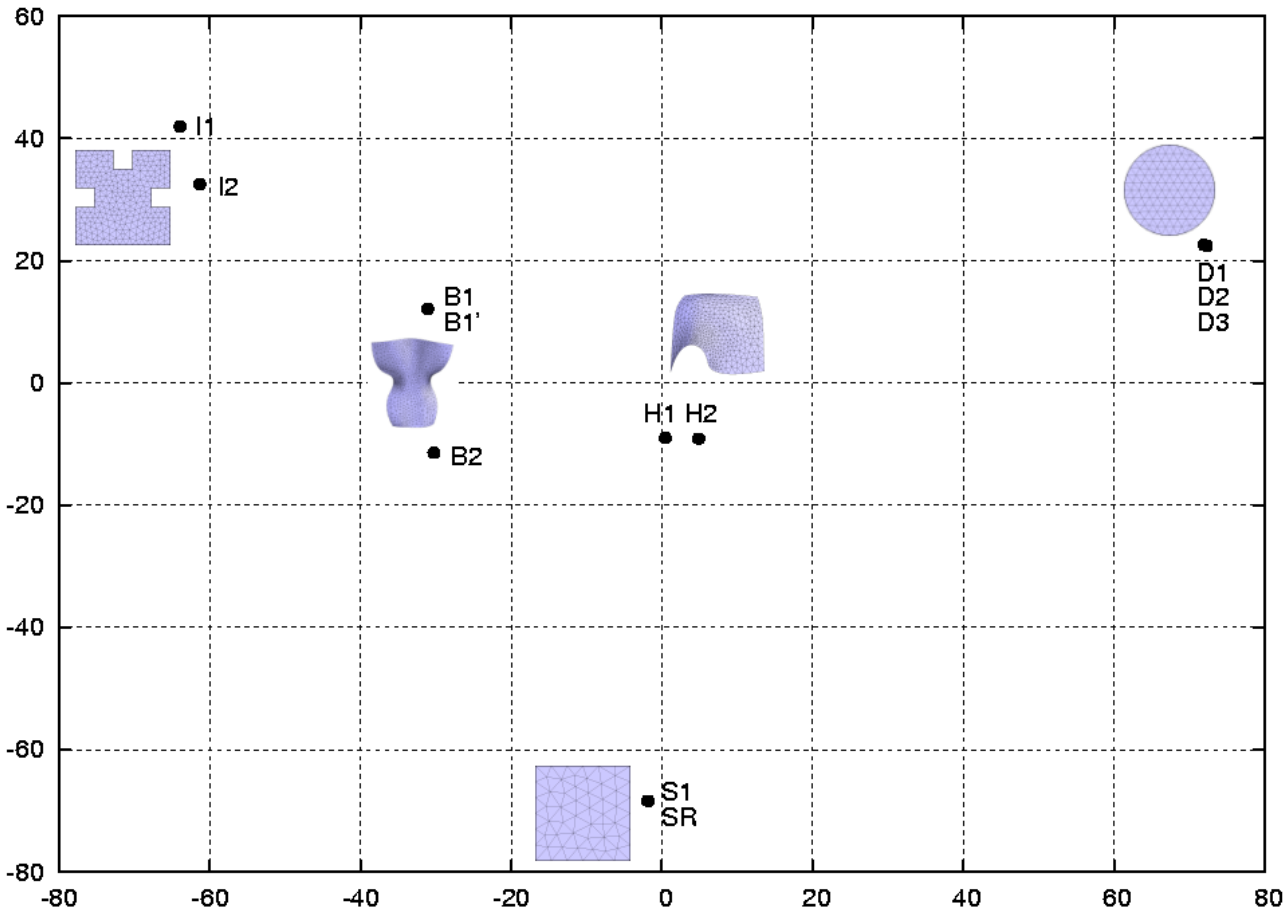
Martin Reuter

Objects of comparison:



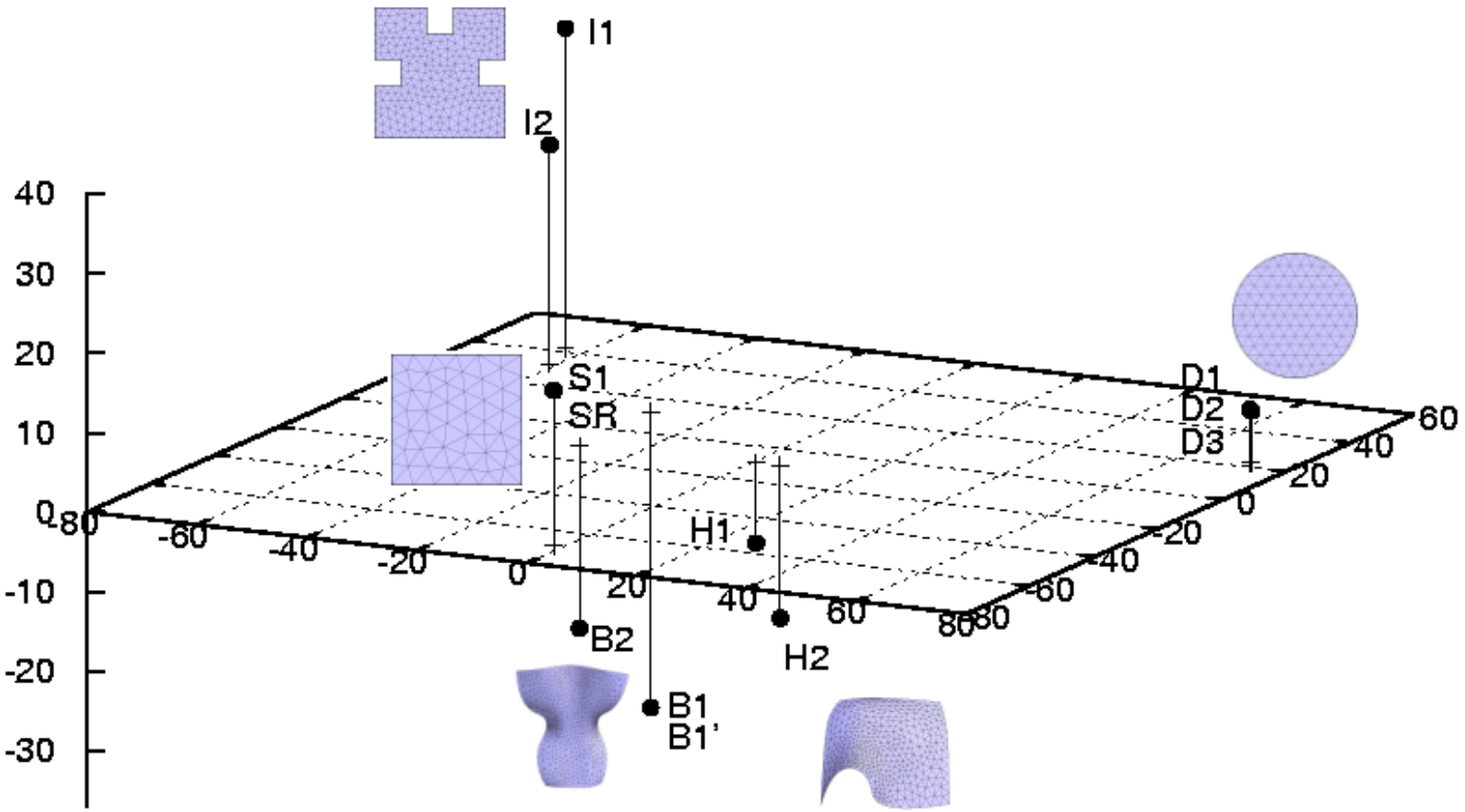
MDS Plot 2D - surface patches

Martin Reuter



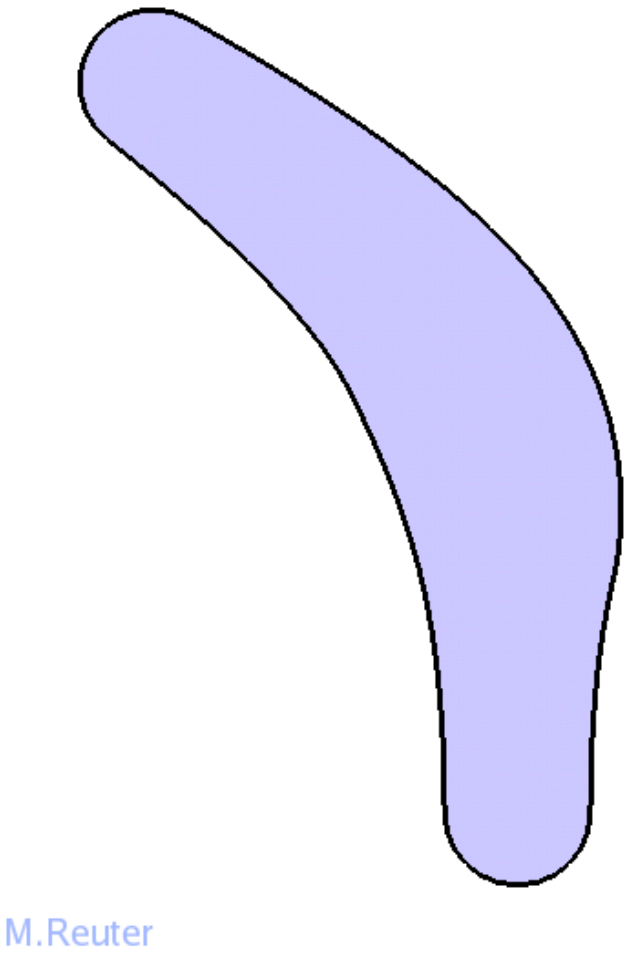
MDS Plot 3D - surface patches

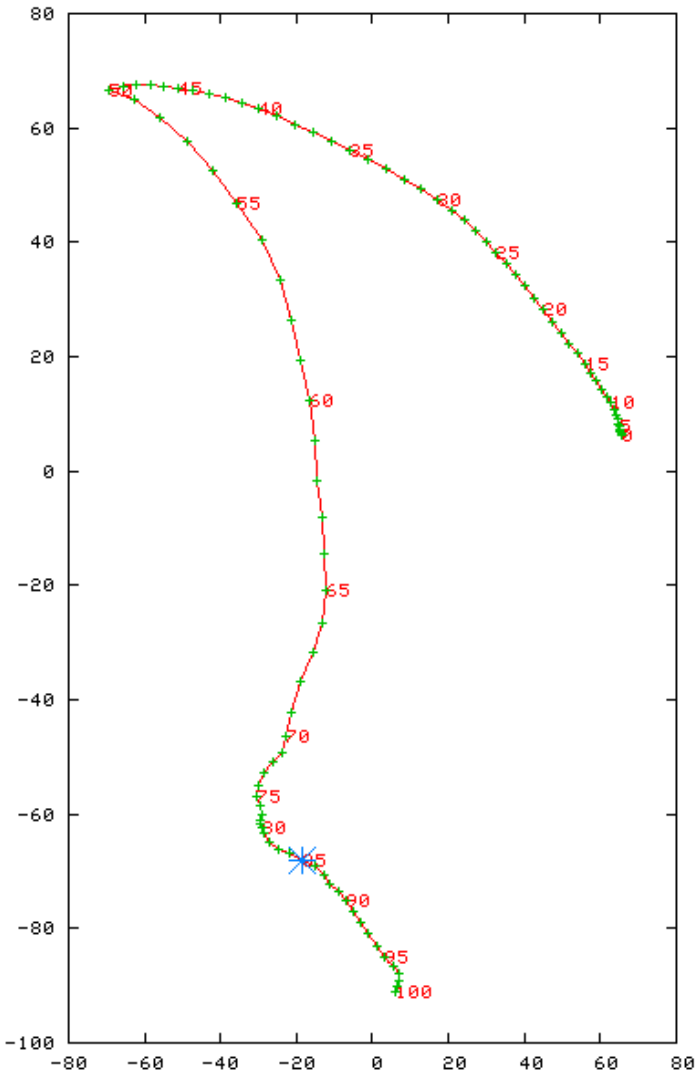
Martin Reuter



MDS Plot 2D - Medial Bar Deformation

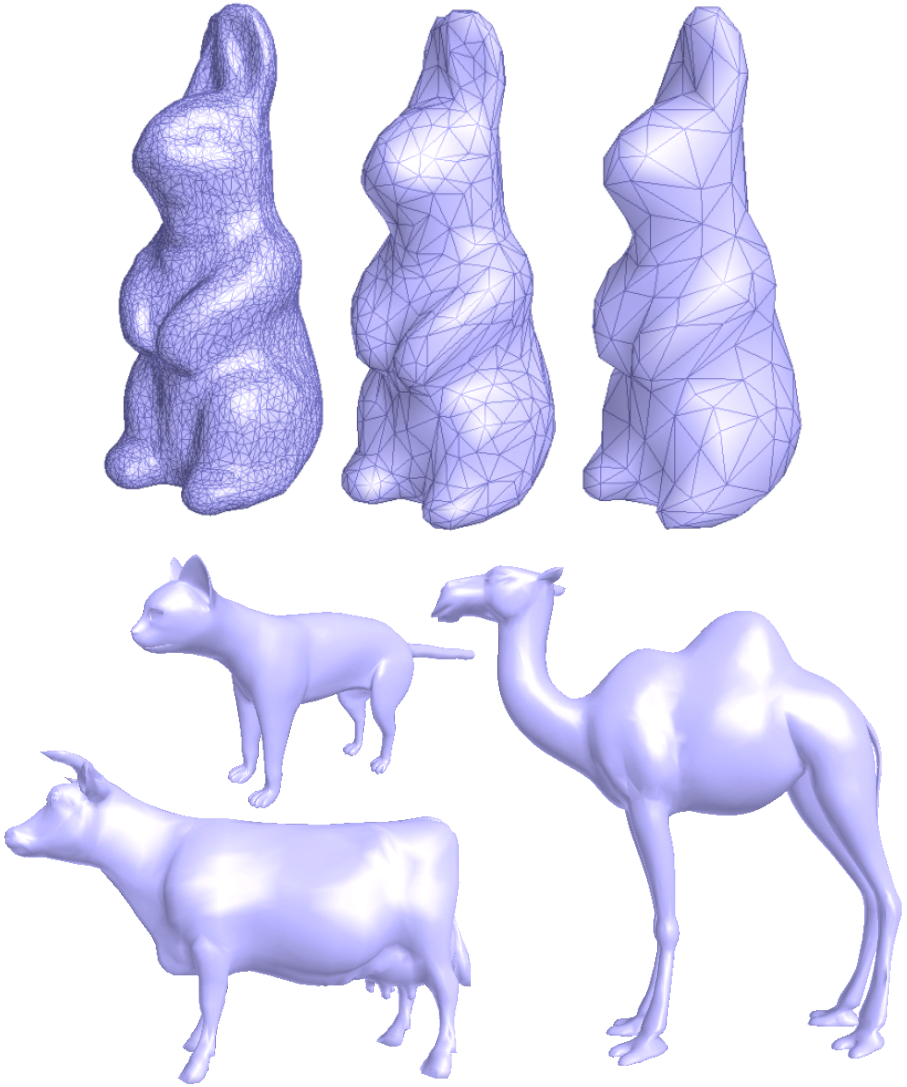
Martin Reuter





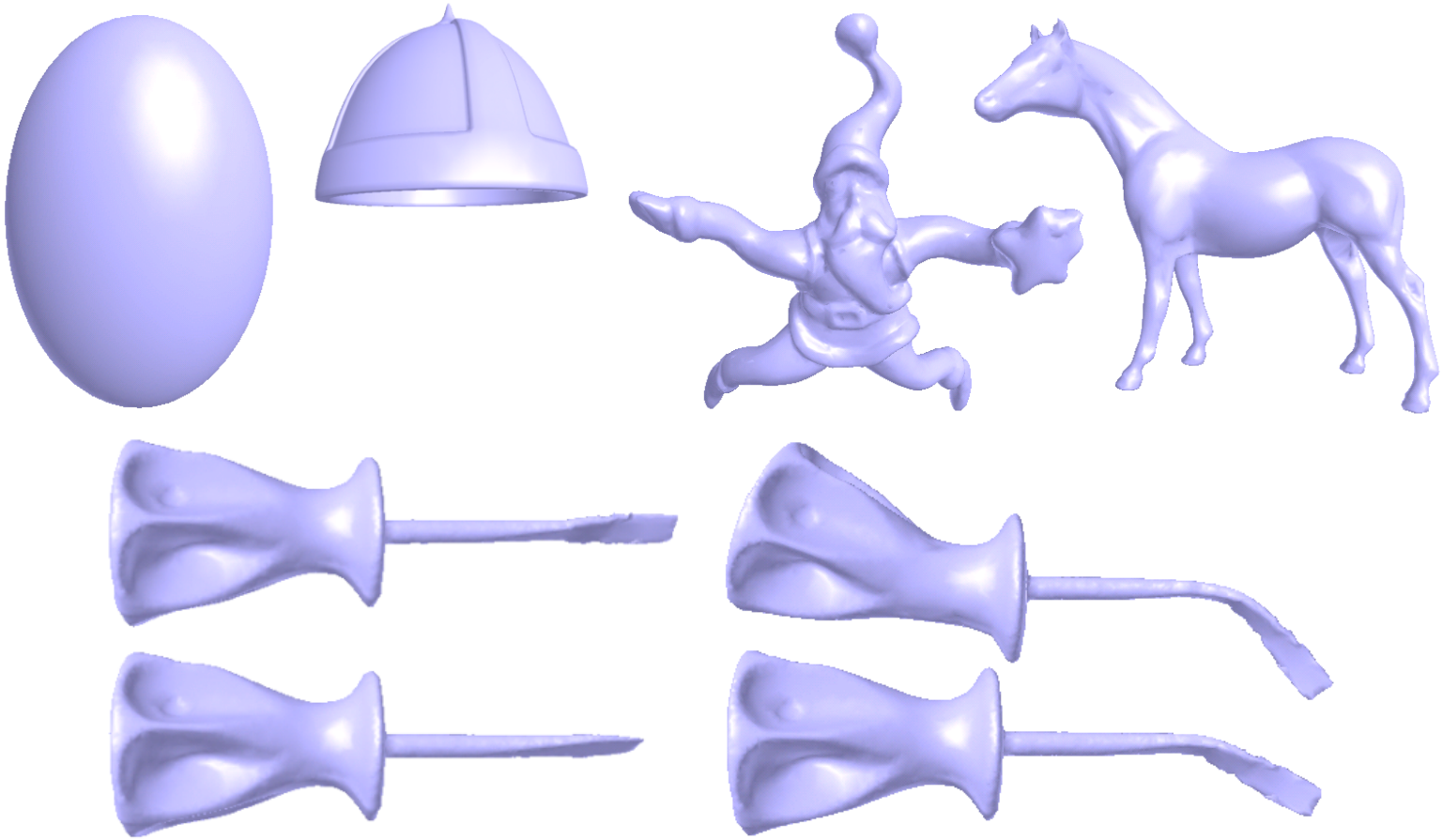
Triangulation of deformed spheres

Martin Reuter



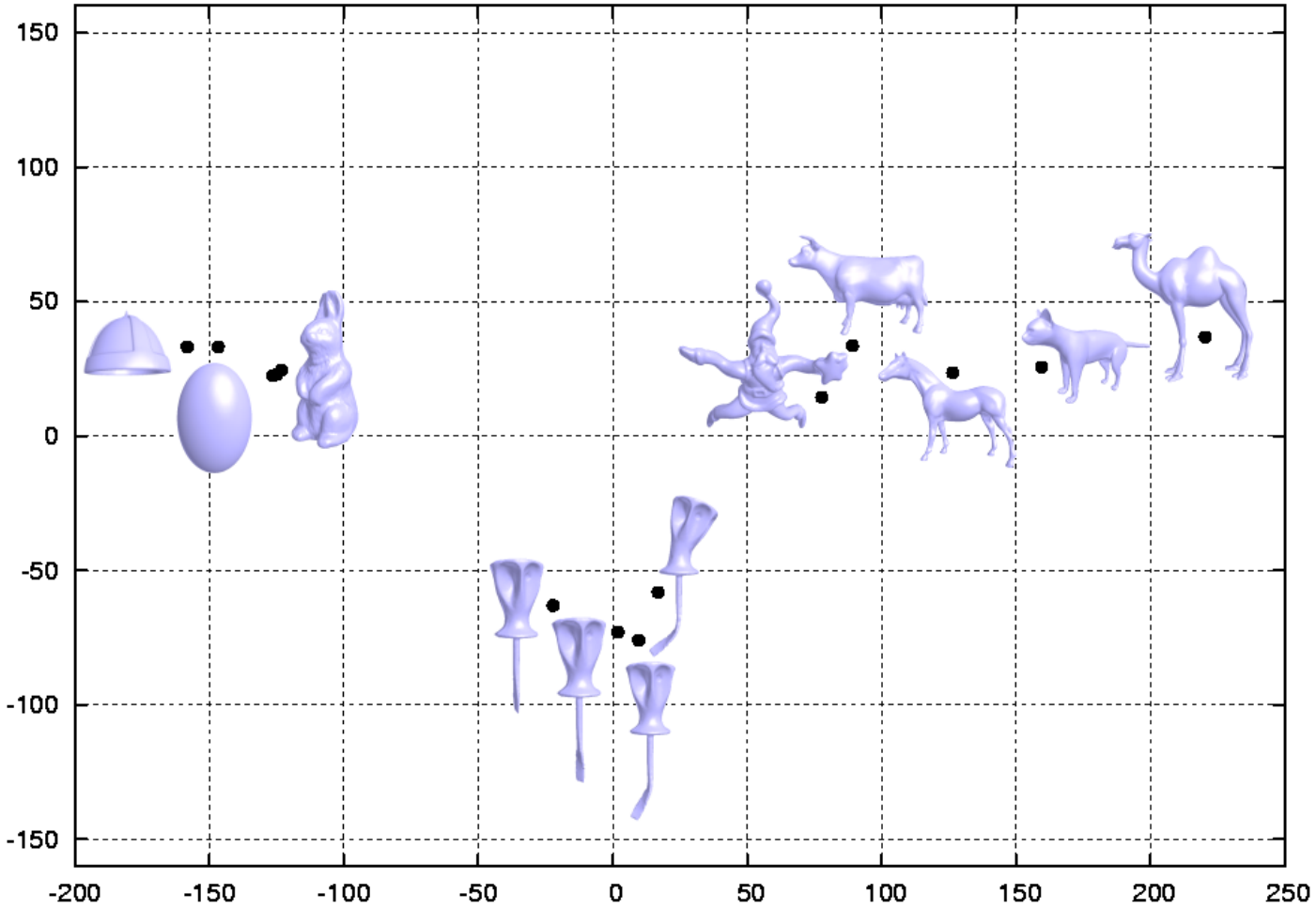
Triangulation of deformed spheres

Martin Reuter



MDS Plot 2D - deformed spheres

Martin Reuter



Global Shape Analysis of Medical Data

Global Shape Analysis on caudate nucleus
Populations (Brain MRI acquired on a 1.5-T General Electric MR scanner):

SPD

32 female subjects diagnosed with Schizotypal Personality Disorder (SPD)

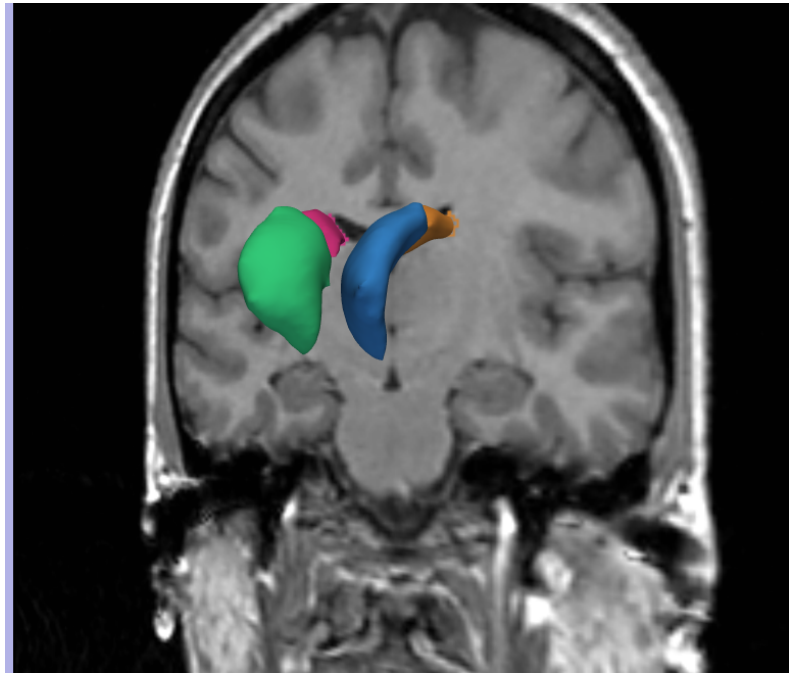
NC

29 female normal control (NC) subjects

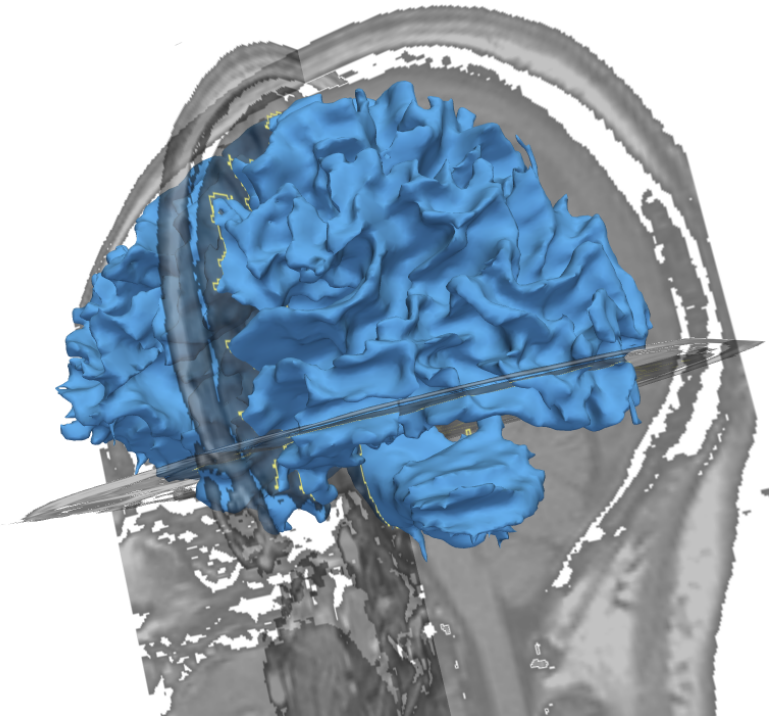
(Harvard Medical - Psychiatry NeuroImaging Laboratory)

Rendering of the Caudate Nucleus

Martin Reuter



Coronal view.

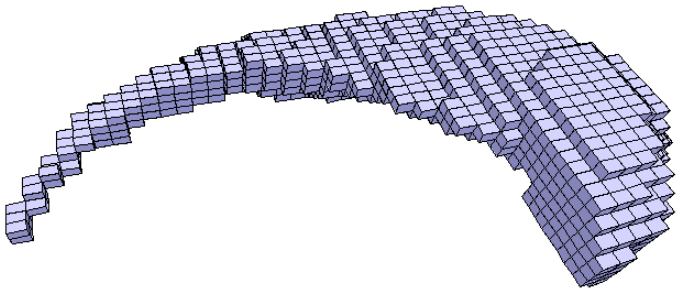


Involved in memory function, emotion processing, and learning.
The caudate nucleus was delineated manually by an expert.

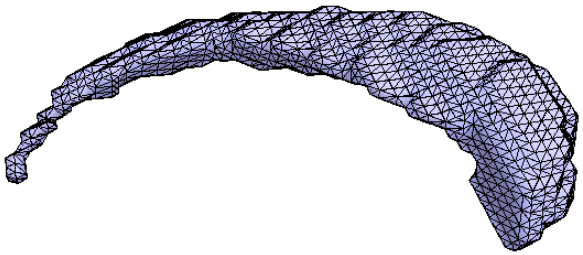
Iso Surfaces from MRI Data

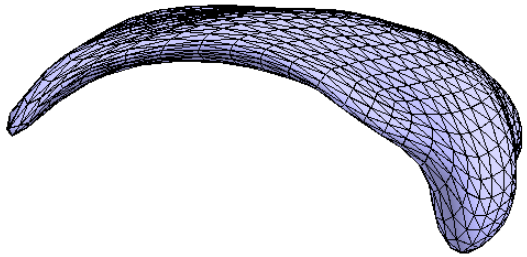
Martin Reuter

Shape comparison either on volumetric data (e.g. tetrahedrization or directly on binary voxel data):



or extraction of (smoothed) iso surfaces:





Statistical Analysis

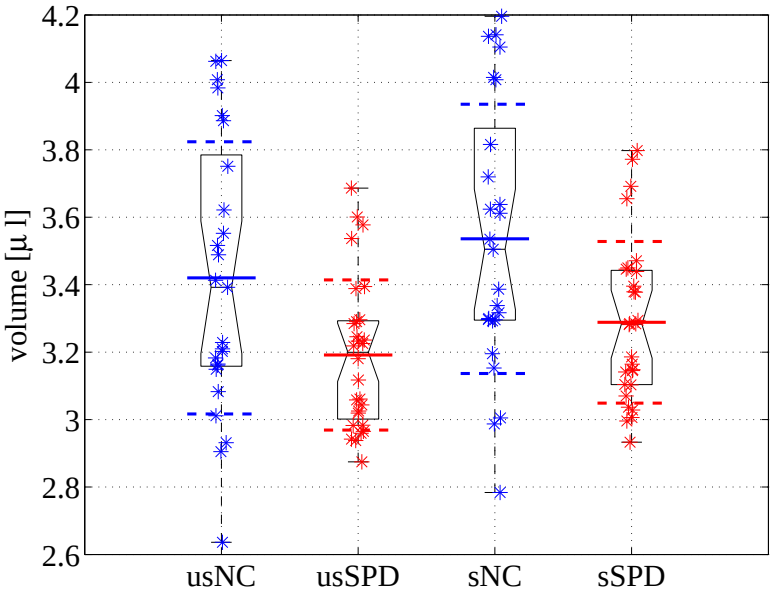
Permutation tests to compare group features (200,000 permutations):

1. Two-sided, nonparametric scalar permutation test for volume and surface area
2. Two-sided, nonparametric multivariate permutation test based on maximum T-statistic for shapeDNA (normalized eigenvalues)
3. Individual permutation test on shapeDNA components to analyze individual significance

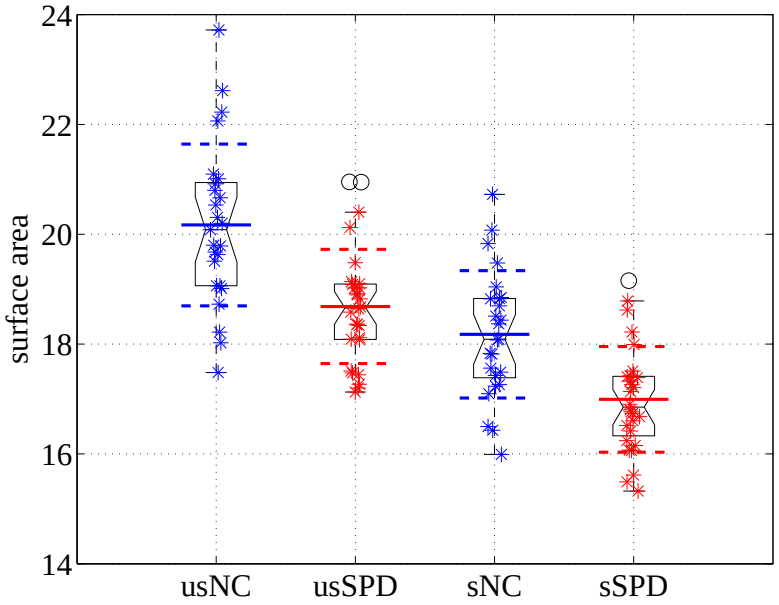
Statistical Analysis

Martin Reuter

Statistically significant volume and surface area reductions of SPD vs. normal control population



volume differences



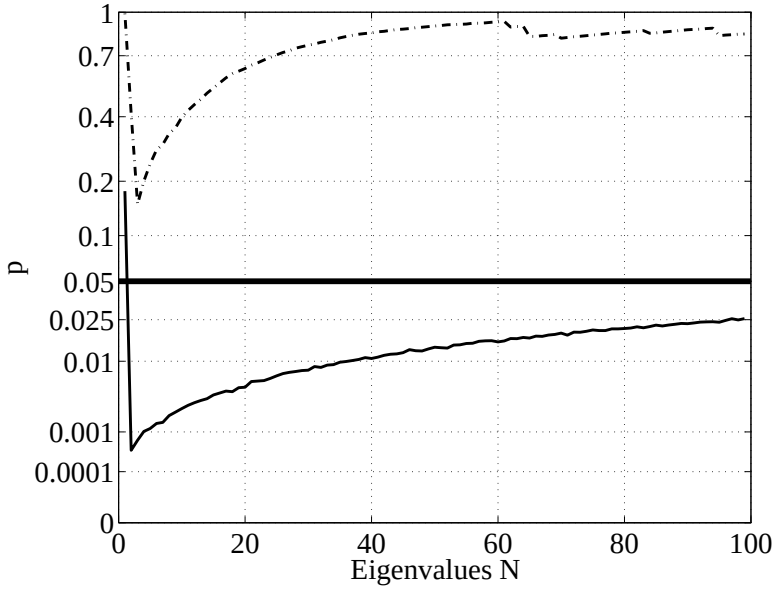
surface area differences

Smoothed results prefix 's', unsmoothed results prefix 'us'.

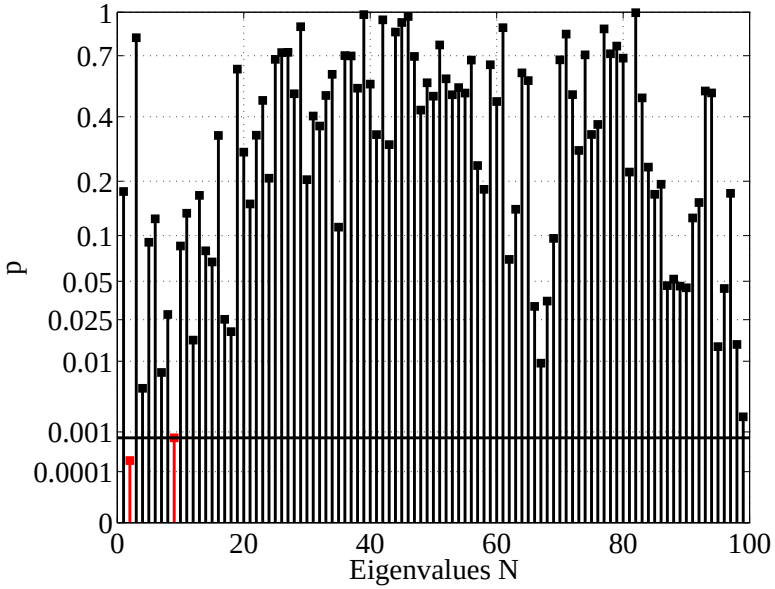
Statistical Analysis 2D

Martin Reuter

Maximum t-statistic results of area **normalized** case (indicating true shape differences)



For the first n eigenvalues



for individual eigenvalues

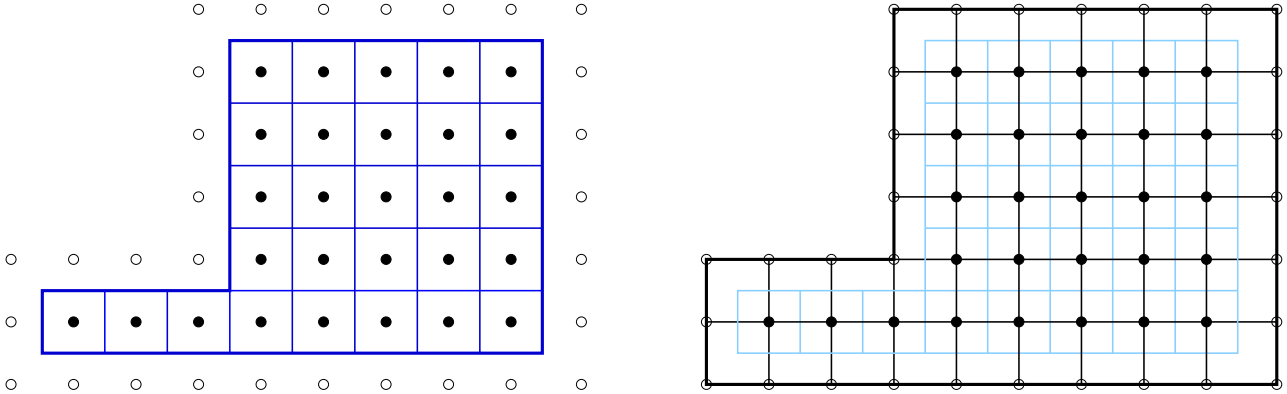
Black horizontal line indicates the 5% significance level.
→ Smoothing leads to information loss.

Statistical Analysis 3D

Martin Reuter

For 3D analysis we worked with

- both Dirichlet and Neumann spectra
- on original voxel domain and dual voxel graph (to introduce more inner nodes).

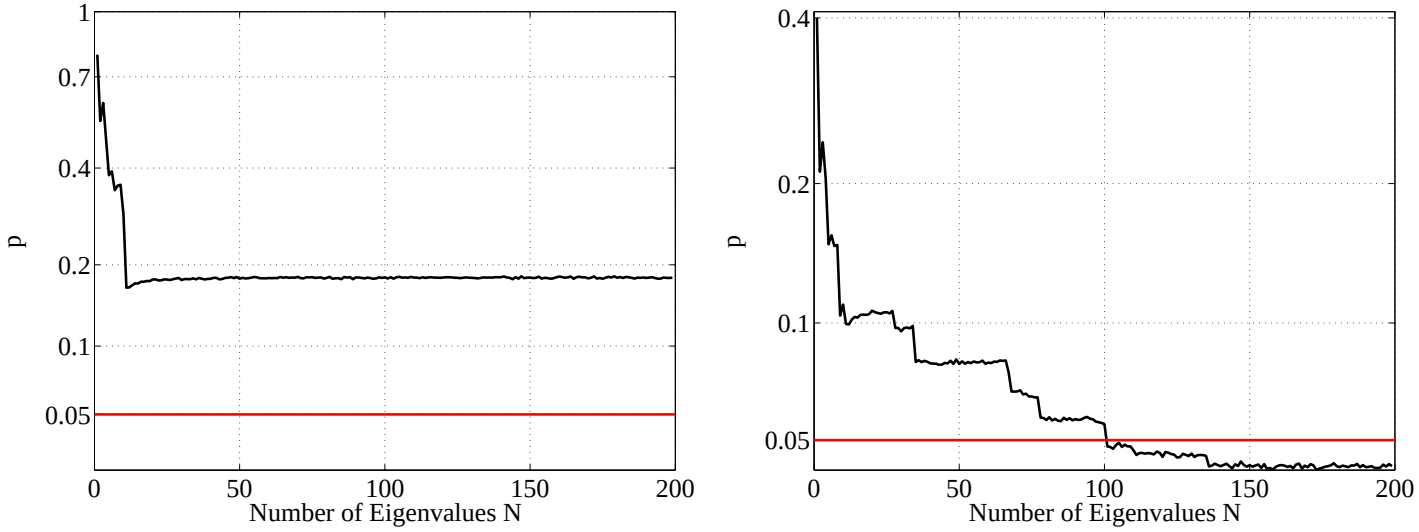


Pixel domain (left) and its dual (right)

Statistical Analysis 3D - Dirichlet

Martin Reuter

Accumulated statistic
(unnormalized Dirichlet regular and dual)

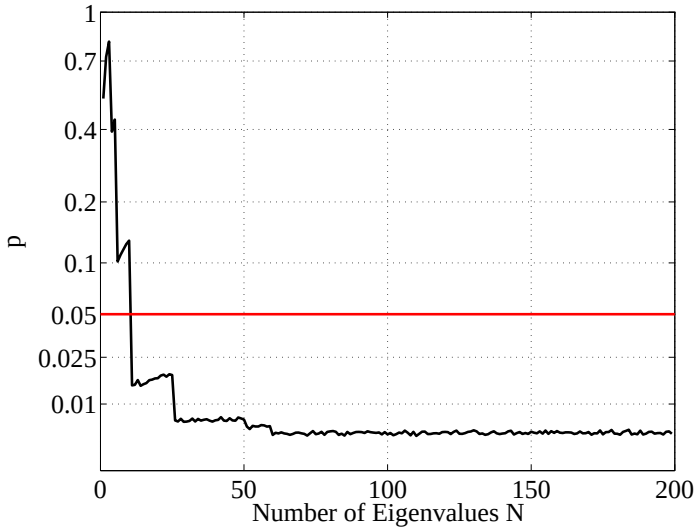


- Left (regular): No statistical significance due to too low resolution
- Right (dual): Statistical significance when high eigenvalues are involved.

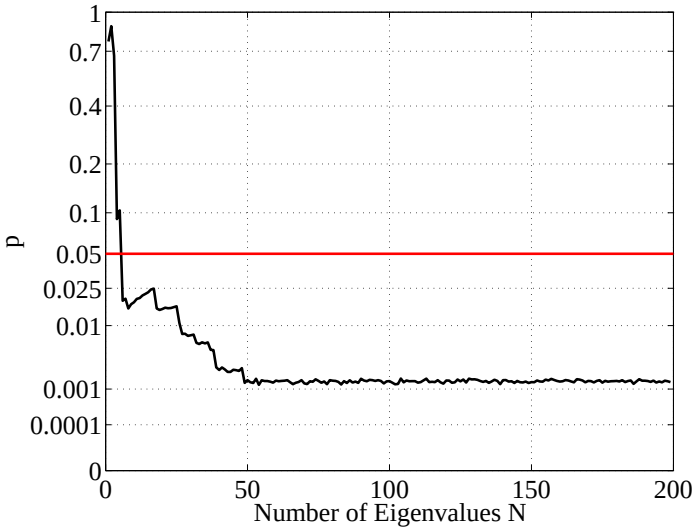
Statistical Analysis 3D - Neumann

Martin Reuter

Neumann Spectra show significant differences very early:



(p) Regular, unit brain volume.



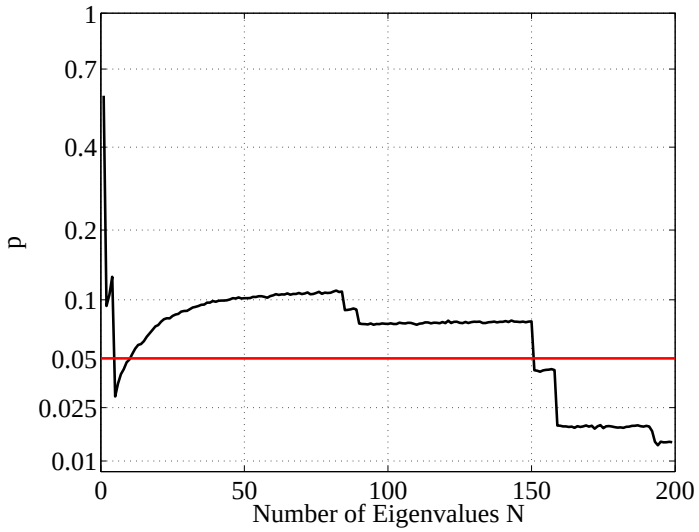
(q) Dual, unit brain volume.

Accumulated statistic, Neumann boundary conditions.

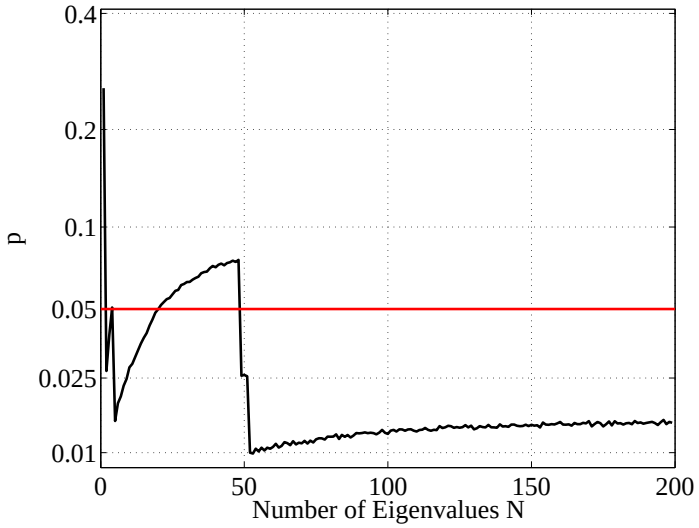
Statistical Analysis 3D - Neumann

Martin Reuter

Checking for true shape differences (unit caudate volume):



(r) Regular, unit caudate volume.



(s) Dual, unit caudate volume.

Accumulated statistic, Neumann boundary conditions.

Conclusion

- Volumetric spectra are applicable for 3D shape analysis
- Neumann spectrum has advantages especially for low resolutions
- Higher Eigenvalues yield significant results, indicating differences in smaller features
- Compare shape based on feature size (multiresolution)

Thanks

Thank you very much for your attention !

Publications can be found at
http://reuter.mit.edu